Optimal Capital Structure with Endogenous Default and Volatility Risk

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Credit Risk Modeling

Credit Risk: the possibility that a counterparty (firm) does not meet its obligations stated in the contract → financial loss (distress)

Modeling:

- Firm Value Models / Structural Models - Merton (1974)

  First Passage Time Models - Black and Cox (1976), Leland (1994), Leland and Toft (1996) ...

  → **Default** / Bankruptcy: exogenous / time-dependent / endogenous failure barrier;

- Intensity Models - Jarrow, Lando, Turnbull (1997), Duffie, Lando (2001) ...

  → **Default** Intensity - Poisson-like Process

→ **Contingent Claim Valuation**

Related Literature

Starting point:

poor job of structural models in predicting credit spreads.

→ introduce jumps and/or to *remove the assumption of constant volatility*.

Chen, Khou (2009); Dao, Jeanblanc (2006) (double exponential jump diffusion); Fiorani, Luciano, Semeraro (2010) (pure jump process of the VG type); Hilberink, Rogers (2002), Hurd (2009) (log-leverage as time changed brownian motion); Longstaff-Schwartz (1995) (stochastic interest rates); Duffie, Lando (2001) (incomplete information); Fouque, Papanicolau, Solna (2005) (stochastic volatility).

→ We **focus** on Leland (1994) model and introduce **volatility risk**
→ influence on: financial variables, endogenous failure level, optimal capital structure; credit spreads - leverage ratios.

Leland (1994) Capital Structure Model

Structural - First Passage Time Model

→ *Contingent Claim Valuation*

- Firm's Assets Value → Underlying Asset Dynamics
  → *GBM - constant volatility*
- Firm's Capital Structure (value) → Derivatives Contracts (price)
- Default Barrier: endogenously derived - equity holders maximizing behavior - optimal stopping problem - smooth pasting condition
- Infinite horizon - Single Debt outstanding

From Leland (1994) Model...

...we **REMOVE**:

$$dV_t = rV_t dt + \sigma V_t dW_t$$

...we **KEEP**:

- infinite time horizon
- single debt D outstanding, paying C per unit of time
- corporate tax rate τ, tax benefits of debt τC
- bankruptcy costs $0 < \alpha < 1$, strict priority rule
- endogenous default

$\rightarrow$ **Derivative Contracts: E** (equity), **D** (debt), **v** (total firm value), **TB** (tax benefits), **BC** (bankruptcy costs)

- payoff at default, payment until default

$\rightarrow$ **coupon-paying defaultable time-independent securities**

Research Idea: Volatility Risk

Analyze the **capital structure of a firm in an infinite time horizon** framework following Leland (1994) under the more general hypothesis that the **firm's assets value process belongs to a fairly large class of stochastic volatility models**, with volatility being driven by **a one factor fast mean reverting process of Orstein-Uhlenbeck type**.

$\downarrow$

Is this framework able to predict:

- ... enhanced credit spreads?
- ... lower leverage ratios?

The Stochastic Volatility *Pricing* Model

Under a *risk neutral* probability measure , the asset's evolution follows the SDEs:

$$dV_t^\epsilon = rV_t^\epsilon dt + f(Y_t^\epsilon)V_t^\epsilon dW_t,\ V_0^\epsilon = x, \quad (1)$$

$$dY_t^\epsilon = \left(\frac{1}{\epsilon}(m - Y_t^\epsilon) - \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}\Lambda(Y_t^\epsilon)\right)dt + \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}d\widetilde{W}_t,\ Y_0^\epsilon = y, \quad (2)$$

$$d\langle W, \widetilde{W}\rangle_t = \rho dt, \quad \rho \in (-1, 0), \quad (3)$$

$$\Lambda(Y_t^\epsilon) := \rho\frac{\mu - r}{f(Y_t^\epsilon)} + \gamma(Y_t^\epsilon)\sqrt{1-\rho^2}, \quad (4)$$

→ μ expected growth rate (physical measure)
→ $\gamma(Y_t^\epsilon)$ market price of risk
→ ϵ: time scale parameter - fast mean-reversion
→ $f(Y_t^\epsilon)$: positive, non-decreasing func., bounded above and away from 0

$$\bar{\sigma}^2 := \int_{\mathbb{R}} f^2(y)\Phi(y)dy, \quad (5)$$

→ $\Phi(y)$ Gaussian density $N(m, \nu^2)$.

Contingent Claim Valuation

Under the risk neutral measure, the price of a **coupon-paying time-independent defaultable claim** in this stochastic volatility model (1)-(2) is:

$$P^{\epsilon}(x,y) = \mathbb{E}\left[e^{-r{T_B}^{\epsilon}} b(x_B) + c \int_0^{{T_B}^{\epsilon}} e^{-rs} ds \,|\, V_0^{\epsilon} = x, Y_0^{\epsilon} = y \right], \qquad (6)$$

where $b(x_B)$ is the payoff at default, c the continuous constant coupon.

Default Barrier.
The first passage time of V_t^{ϵ} at x_B is defined as

$${T_B}^{\epsilon} = \inf\{t \geq 0 : V_t^{\epsilon} = x_B\}, \qquad 0 < x_B < x.$$

$\rightarrow$ **Laplace transform** *of the stopping time* ${T_B}^{\epsilon}$*:* **not available !**

$$\mathbb{E}\left[e^{-r{T_B}^{\epsilon}} | V_0^{\epsilon} = x, Y_0^{\epsilon} = y \right] \qquad ?$$

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$$\mathbb{E}\left[e^{-r{T_B}^{\epsilon}} | V_0^{\epsilon}=x, Y_0^{\epsilon}=y\right] \quad ?$$

$$\rightarrow \quad \widetilde{P}^{\epsilon}(x) := P_0(x) + \sqrt{\epsilon}P_1(x) \rightarrow \quad \mathrm{BS}(\bar{\sigma})$$

Leading Order Term: $P_0(x)$

Proposition. Let $P_{BS}(x;\bar{\sigma})$ be the Black-Scholes price with constant volatility $\bar{\sigma}$ in (5) under an infinite time horizon.

For each **coupon-paying time-independent defaultable claim** the leading order term of its price approximation $P_0(x) := P_{BS}(x;\bar{\sigma})$ (same contract) and can be written under the following general form:

$$P_0(x) = k(x) + \left(b(x_B) - \frac{c}{r}\right)\left(\frac{x_B}{x}\right)^{\lambda}, \quad (7)$$

with

$$\mathbb{E}[e^{-rT_B}|V_0 = x] = \left(\frac{x_B}{x}\right)^{\lambda}, \quad \lambda = 2r/\bar{\sigma}^2,$$

→ $k(x)$: default-free part of the Black-Scholes price
→ c : constant continuous coupon paid by the claim
→ $b(x_B)$: payoff of the claim at default.

First Order Correction Term: $P_1(x)$

Proposition. Under the stochastic volatility model (1)-(2), the first-order fast scale correction terms $P_1(x)$ have the following general structure:

$$P_1(x) = \left(b(x_B) - \frac{c}{r}\right) H(\rho, \bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^{\lambda} \log \frac{x}{x_B}, \quad (8)$$

with

$$\lambda = 2r/\bar{\sigma}^2 \quad (9)$$

$$H(\rho, \bar{\sigma}) = \frac{4r}{\bar{\sigma}^4}\left(\frac{\nu}{\sqrt{2}}\langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2} v_3\right), \quad (10)$$

$$v_3 = \frac{\rho}{\sqrt{2}} \nu \langle f \phi' \rangle, \quad (11)$$

$$\phi'(y) := \frac{1}{\nu^2 \Phi(y)} \int_{-\infty}^{y} \left(f(z)^2 - \bar{\sigma}^2\right) \Phi(z) dz, \quad (12)$$

$$\langle g \rangle := \int_{\mathbb{R}} g(y) \Phi(y) dy, \quad (13)$$

with $\Lambda(\cdot)$, $\bar{\sigma}^2$ given in (4), (5), $\rho \in (-1, 0)$, $r, \nu > 0$,
$\Phi(y)$ Gaussian density $N(m, \nu^2)$, $f(\cdot)$ vol. process.

Volatility Risk Correction

(Idea) $\rightarrow$ The *corrected* price

$$\widetilde{P}^{\epsilon}(x) := P_0(x) + \sqrt{\epsilon} P_1(x) \tag{14}$$

is solution of

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0(x) + \sqrt{\epsilon} P_1(x)) = V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}, \tag{15}$$

with

$$V_2 = \sqrt{\epsilon} v_2 = \sqrt{\epsilon} \left(2 v_3 - \frac{\sqrt{2}}{2} \nu \left\langle \Lambda \phi' \right\rangle \right), \quad \rightarrow \sigma^* = \sqrt{\bar{\sigma}^2 - 2 V_2}$$

$$V_3 = \sqrt{\epsilon} v_3 = \sqrt{\epsilon} \rho \frac{\sqrt{2}}{2} \nu \langle f \phi' \rangle, \quad \rightarrow \textit{skew},$$

$\rightarrow \Lambda, \phi'(\cdot), \langle \cdot \rangle$ given in (4), (12), (13), $\nu > 0, \rho \in (-1, 0)$.

$$\lim_{\mathbf{x} \to \infty} \frac{\mathbf{P_0(x)}}{\mathbf{x}} < \infty, \quad \mathbf{P_0(x_B) = b(x_B)}, \quad (\mathbf{BC : P_0})$$

$$\lim_{\mathbf{x} \to \infty} \mathbf{P_1(x) = 0}, \quad \mathbf{P_1(x_B) = 0}, \quad (\mathbf{BC : P_1}).$$

Pricing *Capital Structure* Claims under Volatility Risk

Proposition. Each defaultable claim on V_t^ϵ has the following structure:

$$\widetilde{P}^\epsilon(x) = \underbrace{P^{L,\bar{\sigma}}(x)}_{BS} + \sqrt{\epsilon}\left(b(x_B) - \frac{c}{r}\right) H(\rho,\bar{\sigma}) \left(\frac{x_B}{x}\right)^\lambda \log\frac{x}{x_B}, \tag{16}$$

or equivalently

$$\widetilde{P}^\epsilon(x) := k(x) + \left(b(x_B) - \frac{c}{r}\right)\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}), \tag{17}$$

with

$$h_\epsilon(x, x_B; \rho, \bar{\sigma}) := 1 + \sqrt{\epsilon} H(\rho,\bar{\sigma}) \log\frac{x}{x_B} \tag{18}$$

being a *DEFAULT-dependent VOLATILITY RISK correction* for the price.

Leland (1994): Capital Structure Claims Value

Under Leland (1994) setting with constant *effective volatility* $\bar{\sigma}$, the capital structure defaultable claims are

$$E^{L,\bar{\sigma}}(x) = x - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - x_B\right)\left(\frac{x_B}{x}\right)^{\lambda}, \qquad (19)$$

$$D^{L,\bar{\sigma}}(x) = \frac{C}{r} + \left((1-\alpha)x_B - \frac{C}{r}\right)\left(\frac{x_B}{x}\right)^{\lambda}, \qquad (20)$$

$$BC^{L,\bar{\sigma}}(x) = \alpha x_B \left(\frac{x_B}{x}\right)^{\lambda}, \qquad (21)$$

$$TB^{L,\bar{\sigma}}(x) = \frac{\tau C}{r} - \frac{\tau C}{r}\left(\frac{x_B}{x}\right)^{\lambda}, \qquad (22)$$

$$v^{L,\bar{\sigma}}(x) = x + \frac{\tau C}{r} - \left(\frac{\tau C}{r} + \alpha x_B\right)\left(\frac{x_B}{x}\right)^{\lambda}, \qquad (23)$$

with

$$\mathbb{E}[e^{-rT_B} | V_0 = x] = \left(\frac{x_B}{x}\right)^{\lambda}, \quad \lambda = 2r/\bar{\sigma}^2.$$

Corrected Capital Structure Claims Value

Proposition. Under the stochastic volatility model (1)-(2) the capital structure defaultable claims are

$$\widetilde{E}^{\epsilon}(x) = x - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - x_B\right)\left(\frac{x_B}{x}\right)^{\lambda} h_{\epsilon}(x, x_B; \rho, \bar{\sigma}),$$

$$\widetilde{D}^{\epsilon}(x) = \frac{C}{r} + \left((1-\alpha)x_B - \frac{C}{r}\right)\left(\frac{x_B}{x}\right)^{\lambda} h_{\epsilon}(x, x_B; \rho, \bar{\sigma}), \qquad (24)$$

$$\widetilde{BC}^{\epsilon}(x) = \alpha x_B \left(\frac{x_B}{x}\right)^{\lambda} h_{\epsilon}(x, x_B; \rho, \bar{\sigma}), \qquad (25)$$

$$\widetilde{TB}^{\epsilon}(x) = \frac{\tau C}{r} - \frac{\tau C}{r}\left(\frac{x_B}{x}\right)^{\lambda} h_{\epsilon}(x, x_B; \rho, \bar{\sigma}), \qquad (26)$$

$$\widetilde{v}(x)^{\epsilon} = x + \frac{\tau C}{r} - \left(\frac{\tau C}{r} + \alpha x_B\right)\left(\frac{x_B}{x}\right)^{\lambda} h_{\epsilon}(x, x_B; \rho, \bar{\sigma}), \qquad (27)$$

with the *DEFAULT-dependent VOLATILITY RISK correction*

$$h_{\epsilon}(x, x_B; \rho, \bar{\sigma}) := 1 + \sqrt{\epsilon} H(\rho, \bar{\sigma}) \log \frac{x}{x_B}.$$

Corrected Equity Claim Value

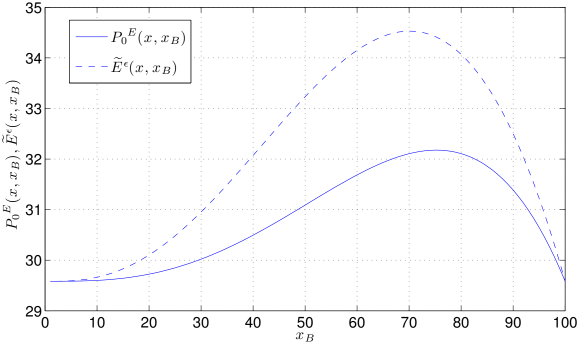

Figure: *Corrected* **Equity Claim Value.** The plot shows corrected equity claim value $\widetilde{E}(x, x_B)$ and ${P_0}^E(x, x_B)$ term as function of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $\rho = -0.05$, $x = 100$.

Endogenous Failure Level $\widetilde{x_B}^{\epsilon}$

Equity holders face the following **optimal stopping problem**:

$$\max_{x_B \in [\bar{x_B}^{\epsilon}, \frac{(1-\tau)C}{r}]} \widetilde{E}^{\epsilon}(x; x_B), \tag{28}$$

which is *not*-**equivalent to apply standard smooth-fit principle**:

$$\frac{\partial \widetilde{E}^{\epsilon}(x; x_B)}{\partial x}|_{x=x_B} = 0 \rightarrow \bar{x_B}^{\epsilon}.$$

Under **volatility risk** the following holds:

$$\bar{x_B}^{\epsilon} < \widetilde{x_B}^{\epsilon} < x_{B\,L}.$$

Corrected Smooth-Pasting Condition

Proposition. *Corrected Smooth-Pasting.*

The endogenous failure level $\widetilde{x_B}^{\epsilon}$ satisfies the following **'corrected smooth-pasting'** condition:

$$\frac{\partial {P_0}^E(x)}{\partial x}\Big|_{x=x_B} h_\epsilon(x, x_B; \rho, \bar{\sigma}) + \sqrt{\epsilon}\frac{\partial {P_1}^E(x)}{\partial x}\Big|_{x=x_B} = 0, \qquad (29)$$

$$h_\epsilon(x, x_B; \rho, \bar{\sigma}) := 1 + \sqrt{\epsilon} H(\rho, \bar{\sigma}) \log \frac{x}{x_B},$$

$$\begin{aligned} {P_0}^E(x) &= E^{L,\bar{\sigma}}(x) \quad \text{in (19)} \\ {P_1}^E(x) &= \left(\frac{(1-\tau)C}{r} - x_B\right) H(\rho, \bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^{\lambda} \log \frac{x}{x_B} \qquad (30) \end{aligned}$$

→ **Smooth-pasting condition failure**: Alili, Kyprianou (2005); Barrieu, Bellamy (2007); Medvedev, Scaillet (2010), etc...

Optimal *Corrected* Capital Structure

Optimal Coupon $\widetilde{\mathbf{C}}^{\epsilon *}$:

$$\max_{\mathbf{C}} \quad \widetilde{\mathbf{v}}^{\epsilon}(\mathbf{x}, \widetilde{\mathbf{x_B}}^{\epsilon}, \mathbf{C}) \quad \rightarrow (\widetilde{x_B}^{\epsilon *}, \widetilde{C}^{\epsilon *})$$

→ Optimal **Credit Spreads:** $\widetilde{\mathbf{R}}^{\epsilon *} - \mathbf{r}$,
→ Optimal **Leverage Ratios:** $\widetilde{\mathbf{L}}^{\epsilon *}$.

$$\begin{aligned} \widetilde{\mathbf{R}}^{\epsilon *} - \mathbf{r} &:= \widetilde{C}^{\epsilon *} / \widetilde{D}^{\epsilon *} - r \\ \widetilde{\mathbf{L}}^{\epsilon *} &:= \widetilde{D}^{\epsilon *} / \widetilde{v}^{\epsilon *} \end{aligned}$$

with

$$\begin{aligned} \widetilde{D}^{\epsilon *} &= \widetilde{D}^{\epsilon}(x, \widetilde{x_B}^{\epsilon *}, \widetilde{C}^{\epsilon *}) \\ \widetilde{v}^{\epsilon *} &= \widetilde{v}^{\epsilon}(x, \widetilde{x_B}^{\epsilon *}, \widetilde{C}^{\epsilon *}) \\ \widetilde{E}^{\epsilon *} &= \widetilde{E}^{\epsilon}(x, \widetilde{x_B}^{\epsilon *}, \widetilde{C}^{\epsilon *}) \end{aligned}$$

Optimal *Corrected* Capital Structure - (Skew)

ρ	$\widetilde{C}^{\epsilon *}$	$\widetilde{D}^{\epsilon *}$	$\widetilde{R}^{\epsilon *} - r$ (bps)	$\widetilde{E}^{\epsilon *}$	$\widetilde{v}^{\epsilon *}$	$\widetilde{L}^{\epsilon *}$ (%)
0	6.50	96.27	75.25	32.16	128.44	74.95 %
-0.05	5.91	84.15	102.19	39.58	123.73	68.01 %
-0.06	5.74	81.45	104.86	41.41	122.87	66.29 %
-0.07	5.57	78.79	106.74	43.25	122.05	64.56 %
-0.08	5.39	76.23	107.99	45.05	121.28	62.85 %
-0.09	5.23	73.79	108.74	46.78	120.57	61.19 %
-0.1	5.20	73.39	108.83	47.06	120.46	60.92 %

Table: **Skew effect on optimal capital structure.** The table shows financial variables at their optimal level when only the *skew effect* is considered, i.e. $\rho \in (-1, 0), \Lambda = 0$. The first row of the table reports Leland (1994) results as benchmark, as particular case of $\rho = 0, \Lambda = 0$. We consider $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$. Recall $V_3 := \sqrt{\epsilon}\rho\frac{\sqrt{2}}{2}\nu\langle f\phi'\rangle$. We consider $V_3 = -0.06\rho$, $V_2 = 2V_3$, see also Fouque et al. (2000), Fouque et al. (2005).

Optimal *Corrected* Capital Structure - (Skew and Vol. Level Correction)

σ^*	$\widetilde{C}^{\epsilon *}$	$\widetilde{D}^{\epsilon *}$	$\widetilde{R}^{\epsilon *} - r$ (bps)	$\widetilde{E}^{\epsilon *}$	$\widetilde{v}^{\epsilon *}$	$\widetilde{L}^{\epsilon *}$ (%)
$\bar{\sigma} = 0.2$	6.50	96.27	75.25	32.16	128.44	74.95 %
$\bar{\sigma} + 0.01$	5.70	80.26	110.25	42.95	123.26	65.12 %
$\bar{\sigma} + 0.02$	5.59	77.35	123.59	41.88	124.04	62.35 %

Table: **Skew effect and volatility level correction: influence on optimal capital structure.** The table shows financial variables at their optimal level when $\rho = -0.05$ and also a volatility correction is considered. Recall that $\sigma^* = \sqrt{\bar{\sigma}^2 - 2V_2}$. We consider $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $V_3 = 0.003$. L^*, R^* are in percentage (%), $R^* - r$ in basis points (bps).

Concluding Remarks

Under volatility risk...

- ...the value of each claim must be *corrected* due to randomness in the riskiness of the firm
- ...no-equivalence between smooth-fit principle and optimal stopping problem solution $\rightarrow$ *corrected* smooth pasting condition
- ...(*skew effect* and *volatility level correction*) credit spreads are higher, despite lower leverages.

$\rightarrow$ *Future Research:*

- ...time scale ϵ effect on optimal capital structure
- ...volatility risk effect on default probability.

Thank You !

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